CHAPTER 5

SPECIAL FUNCTIONS

Special functions are particular mathematical functions which have more or less established names and notations due to their importance in mathematical analysis, functional analysis, physics, or other applications.

Gamma function

Many important functions in applied sciences are defined via improper integrals. Maybe the most famous among them is the Gamma Function.

The gamma function is defined by

$$\Gamma(x)=\int_0^{\infty} e^{-t}\, t^{x-1}\, dt \qquad (1)$$

Let $t=s^2$, then $dt=2s\,ds$. So by substituting in eq. (1) we have

$$\Gamma(x)=2\int_0^{\infty} e^{-s^2}\, s^{2x-1}\, ds \qquad (2)$$

EXAMPLE 1►

Prove that

$(i)\quad \Gamma(1)=1$. $(ii)\quad \Gamma(2)=1$

SOLUTION

(i) Form eq. (1), $\Gamma(1)=\int_0^{\infty} e^{-t}\, dt=\left[-e^{-t}\right]_0^{\infty}=1$.

(ii) $\Gamma(2)=\int_0^{\infty} e^{-t}\, t\, dt$ (using integration by parts)

CamScanner

$$u = t \quad , \quad dv = e^{-t}\,dt$$

$$du = dt \quad , \quad v = -e^{-t}$$

$$\Gamma(2) = \left[-te^{-t}\right]_0^{\infty} + \int_0^{\infty} e^{-t}\,dt = 0 + \left[-e^{-t}\right]_0^{\infty} = 1$$

EXAMPLE 2►

Prove that $\Gamma(x+1) = x\,\Gamma(x)$ (3)

PROOF:

$$\Gamma(x+1) = \int_0^{\infty} e^{-t}\,t^{x}\,dt \qquad \text{(using integration by parts)}$$

$$u = t^{x} \quad , \quad dv = e^{-t}\,dt$$

$$du = xt^{x-1}dt \quad , \quad v = -e^{-t}$$

$$\Gamma(x+1) = \left[-t^{x}e^{-t}\right]_0^{\infty} + \int_0^{\infty} e^{-t}t^{x-1}\,dt = \Gamma(x)$$

For any integer n,

$$\begin{aligned}\Gamma(n+1) &= n\,\Gamma(n) = n(n-1)\Gamma(n-1)\\ &= n(n-1)(n-2)\Gamma(n-2)\\ &\vdots\\ &= n(n-1)(n-2) \qquad (2)(1)\Gamma(1)\end{aligned}$$

So, $\Gamma(n+1) = n!$

EXAMPLE3►

Evaluate $\int_0^{\infty} e^{-x^2}\,x^{5.2}\,dx$, given that $(\Gamma(0.1) = 9.5135)$

CamScanner

SOLUTION

Comparing the integral with the integral (2)

$5.2 = 2n - 1 \Rightarrow \quad n = 3.1$. So

$$\int_0^\infty e^{-x^2} x^{5.2}\, dx = \frac{1}{2}\Gamma(3.1) = \frac{1}{2}(2.1)\,\Gamma(2.1)$$

$$= \frac{1}{2}(2.1)(1.1)\Gamma(1.1)$$

$$= \frac{1}{2}(2.1)(1.1)\,(0.1)\Gamma(0.1)$$

$$= \frac{1}{2}(2.1)(1.1)\,(0.1)\,(9.513) = 1.1$$

EXAMPLE 3 ▶

a) $$\frac{\Gamma(6)}{2\Gamma(3)} = \frac{5!}{2 \times 2!} = 30$$

b) $$\frac{\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} = \frac{\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} = \frac{3}{4}$$

c) $$\frac{\Gamma(3)\cdot\Gamma(2.5)}{\Gamma(5.5)} = \frac{2!\,\Gamma(2.5)}{(4.5)(3.5)(2.5)\,\Gamma(2.5)}$$

d) $$\frac{6\Gamma\left(\frac{8}{3}\right)}{5\Gamma\left(\frac{2}{3}\right)} = \frac{(6)\left(\frac{5}{3}\right)\left(\frac{2}{3}\right)\Gamma\left(\frac{2}{3}\right)}{5\Gamma\left(\frac{2}{3}\right)} = \frac{4}{3}$$

CamScanner

EXAMPLE 4►

Find $\int_0^{\infty}\left(1-2x^2\right)^2 e^{-x}\,dx$

SOLUTION

$$I=\int_0^{\infty}\left(1-2x^2\right)^2 e^{-x}\,dx$$

$$=\int_0^{\infty}\left(1-4x^2+4x^4\right)e^{-x}\,dx$$

$$=\int_0^{\infty} e^{-x}\,dx-4\int_0^{\infty}x^2\,e^{-x}\,dx+4\int_0^{\infty}x^4\,e^{-x}\,dx$$

$$=\Gamma(1)-4\Gamma(3)+4\Gamma(5)$$

$$=1-4(2!)+4(4!)=89$$

EXAMPLE 5►

Prove that

$$\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$$

SOLUTION

By using the second formula (2) of gamma function

$$\Gamma(x)=\int_0^{\infty}e^{-t}\,t^{x-1}\,dt$$

$$\Gamma\left(\frac{1}{2}\right)=\int_0^{\infty}t^{-1/2}e^{-t}\,dt$$

Let $t=x^2 \Rightarrow dt=2x\,dx$

CamScanner

$$\Gamma\left(\frac{1}{2}\right)=2\int_0^{\infty}x^{-1}e^{-x^2}\,x\,dx=2\int_0^{\infty}e^{-x^2}\,dx$$

Let $I=\Gamma\left(\frac{1}{2}\right)=2\int_0^{\infty}e^{-x^2}\,dx=2\int_0^{\infty}e^{-y^2}\,dy$.

$$I^2=4\int_0^{\infty}\int_0^{\infty}e^{-(x^2+y^2)}\,dx\,dy$$

Using polar coordinates (r,θ) we have

$$I^2=4\int_0^{\pi/2}\int_0^{\infty}e^{-r^2}\,r\,dr\,d\theta=-2\left[\int_0^{\pi/2}d\theta\right]\left(e^{-r^2}\right)_0^{\infty}$$

$$=-2(0-1)(\pi/2)=\pi$$

$$\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$$

In case of the negative number,

$$\because\ \Gamma(x+1)=x\,\Gamma(x)\qquad\Rightarrow\qquad\Gamma(x)=\frac{\Gamma(x+1)}{x}$$

EXAMPLE 6►

Calculate $\int_0^{\infty}3^{-4x^2}\,dx$

SOLUTION

$$\int_0^{\infty}3^{-4x^2}\,dx=\int_0^{\infty}e^{\ln\left(3^{-4x^2}\right)}\,dx$$

$$=\int_0^{\infty}e^{\ln(3)^{-4x^2}}\,dx=\int_0^{\infty}e^{-4x^2\ln(3)}\,dx$$

Put $u=4x^2\ln 3\ \Rightarrow\ du=8(\ln 3)x\,dx$

CamScanner

$$dx = \frac{1}{8\ln(3)} \frac{2\sqrt{\ln(3)}}{\sqrt{u}} du$$

$$I = \frac{2\sqrt{\ln 3}}{8\ln(3)} \int_0^{\infty} \frac{e^{-u}}{\sqrt{u}} du = \frac{1}{4\sqrt{\ln(3)}} \int_0^{\infty} u^{-1/2} e^{-u} du$$

So, the last integral is now

$$I = \frac{1}{4\sqrt{\ln(3)}} \Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{4\sqrt{\ln(3)}}$$

EXAMPLE7►

Calculate $\int_0^1 \frac{1}{\sqrt{-\ln x}} dx$

SOLUTION

Let $-\ln x = t \quad \Rightarrow \quad x = e^{-t} \quad \Rightarrow \quad dx = -e^{-t} dt$

Limits: *as* $x \to 0^+, \quad t = -\ln x \to \infty, \quad x = 1 \to \quad t = 0$

Thus,

$$\int_0^1 \frac{1}{\sqrt{-\ln x}} dx = -\int_{\infty}^0 \frac{e^{-t}}{\sqrt{t}} dt = \int_0^{\infty} t^{-1/2} e^{-t} dt = \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

EXAMPLE8►

Calculate $\int_0^{\infty} e^{-kt} \sqrt{t}\, dt$

SOLUTION

$$I = \int_0^{\infty} e^{-kt} \sqrt{t}\, dt$$

CamScanner

Let $x = kt \quad \Rightarrow \quad t = x/k \quad \Rightarrow \quad dt = dx/k$

Limits: as $t \to 0, \quad x = 0 \quad and \quad t \to \infty, \quad x \to \infty$

$$I = \int_0^\infty e^{-x} \sqrt{\frac{x}{k}} \frac{dx}{k} = \frac{1}{k^{3/2}} \int_0^\infty x^{1/2} e^{-x}\, dx$$

$$I = \frac{1}{k^{3/2}} \Gamma\left(\frac{3}{2}\right) = \frac{1}{k^{3/2}} \frac{1}{2} \Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2k^{3/2}}$$

EXAMPLE 9▶

Evaluate, $\Gamma\left(-\frac{1}{2}\right)$ and $\Gamma\left(-\frac{5}{2}\right)$

SOLUTION

$$\Gamma\left(-\frac{1}{2}\right) = \frac{\Gamma(1/2)}{-1/2} = -2\sqrt{\pi}$$

$$\Gamma\left(-\frac{5}{2}\right) = \frac{\Gamma(-3/2)}{-5/2} = \frac{\Gamma(-1/2)}{(-3/2)(-5/2)} = -\frac{8}{15}\sqrt{\pi}$$

EXAMPLE 10▶

Prove that $2^n \Gamma\left(n + \frac{1}{2}\right) = (1)(3)(5)\cdots(2n-1)\sqrt{\pi}$

SOLUTION

$$\Gamma\left(n + \frac{1}{2}\right) = \left(n - \frac{1}{2}\right) \Gamma\left(n - \frac{1}{2}\right)$$

$$= \left(n - \frac{1}{2}\right)\left(n - \frac{3}{2}\right)\cdots\left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)$$

Multiple both sides by 2^n

CamScanner

$$2^n \Gamma\left(n+\frac{1}{2}\right) = (2n-1)(2n-3)\cdots(5)(3)(1)\sqrt{\pi}$$

Theorem: The complement formula

$$\Gamma(x)\,\Gamma(1-x) = \frac{\pi}{\sin \pi x} \qquad (4)$$

Where x and $1-x$ are not negative or null integers.

Graph of Gamma Function

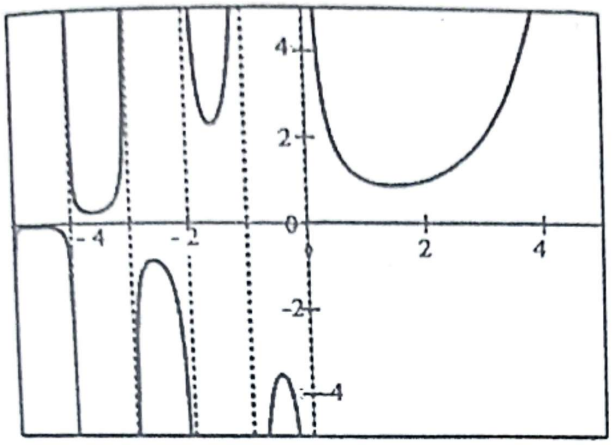
Values of Γ (x) for a range of +ve values of x are available in tabulated form as in table 1.

x	Γ(x)	x	Γ(x)
0.25	3.6256	2.75	1.6084
0.5	1.772	3.00	2.00
0.75	1.2254	3.25	2.5493
1.0	1.00	3.50	3.3234
1.25	0.9064	3.75	4.4230
1.50	0.8862	4.00	6.00
1.75	0.9191	4.25	8.2851
2.00	1.00	4.50	11.6318
2.25	1.1330	4.75	16.5862
2.50	1.3293	5.00	24.00

These together with the results established above , enable us to draw the graph of y = Γ (x)

CamScanner

Gamma Function



An analytic approximation for $\Gamma(z)$, can be obtained by using Stirling's approximation.

$$\Gamma(x) \square x^{x-\frac{1}{2}} e^{-x} \sqrt{2\pi}\left(1+\frac{1}{12x}+\frac{1}{288x^2}-\frac{139}{51840x^3}-\frac{571}{2488320x^4}\right)$$

Thus, for any real $x > 0$, the function $\Gamma(x)$ interpolates continuously betweensuccessive values of $n!$, and so generalizes the factorial function to non-integralvalues of n. For obvious reasons the gamma function is sometimes called the**factorial function**.

II♦BETA FUNCTION►

Another definite integral which is related to the Γ function is the Beta function $\beta(p,q)$ which is defined as

$$\beta(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}\,dx \qquad (5)$$

There are properties and many other forms of Beta function

CamScanner

THEOREM 1 ▶

$$(i)\quad \beta(m,n)=\beta(n,m) \tag{6}$$

$$(ii)\quad \beta(m,n)=2\int_0^{\frac{\pi}{2}}\sin^{2m-1}\theta\cos^{2n-1}\theta\,d\,\theta \tag{7}$$

PROOF

i] Let $x=1-y \quad\Rightarrow\quad dx=-dy.$

At $x=0 \quad\Rightarrow\quad 0=1-y \quad\Rightarrow\quad y=1$

Similarly, at $x=1 \quad\Rightarrow\quad y=0$

Substituting in eq. (5)

$$\beta(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

$$=\int_1^0 (1-y)^{m-1}y^{n-1}\,(-dy)$$

$$=\int_0^1 y^{n-1}(1-y)^{m-1}\,dy$$

$$=\beta(n,m)$$

ii] Let $x=\sin^2\theta \quad\Rightarrow\quad dx=2\sin\theta\cos\theta\,d\theta.$

At $x=0 \quad\Rightarrow\quad 0=\sin^2\theta \quad\Rightarrow\quad \theta=0$

At $x=1 \quad\Rightarrow\quad 1=\sin^2\theta \quad\Rightarrow\quad \theta=\frac{\pi}{2}$

Substituting in eq. (5)

$$\beta(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

CamScanner

$$= \int_0^{\frac{\pi}{2}} \sin^{2m-2}\theta \cos^{2m-2}\theta \cdot 2\sin\theta\cos\theta \, d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2m-1}\theta \, d\theta$$

The following theorem represents the relation between Beta and Gamma functions

THEOREM 2▶

$$\beta(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)} \quad (8)$$

PROOF:

By using the second formula for Gamma function (eq. (2)).

$$\Gamma(m) = 2\int_0^{\infty} e^{-x^2} x^{2m-1} \, dx$$

$$\Gamma(n) = 2\int_0^{\infty} e^{-y^2} y^{2n-1} \, dy$$

$$\Gamma(m)\,\Gamma(n) = 4\int_0^{\infty}\int_0^{\infty} x^{2m-1} y^{2n-1} e^{-\left(x^2+y^2\right)} \, dx \, dy$$

Changing into polar coordinates

$$\Gamma(m)\,\Gamma(n) = 4\int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{\infty} (r\cos\theta)^{2m-1} (r\cos\theta)^{2n-1} e^{-r^2} \, dr\, d\theta$$

$$= \left(2\int_0^{\infty} r^{2(m+n)-1} e^{-r^2} \, dr\right)\left(2\int_0^{\pi/2} \cos^{2m-1}\theta \sin^{2n-1}\theta \, d\theta\right)$$

CamScanner

$$= \Gamma(m+n)\ \beta(n, m)$$

EXAMPLE 1►

$$\int_0^1 x^4 (1-x)^4\ dx = \beta(5, 4)$$

$$= \frac{\Gamma(5)\ \Gamma(4)}{\Gamma(9)} = \frac{(4!)\ (3!)}{8!} = \frac{1}{280}$$

EXAMPLE2►

Evaluate $\int_0^2 \frac{x^2\ dx}{\sqrt{2-x}}$

SOLUTION

Let $x = 2y \quad \Rightarrow \quad dx = 2\ dy$

At $x = 0 \quad \Rightarrow \quad 0 = 2y \quad \Rightarrow \quad y = 0$

At $x = 2 \quad \Rightarrow \quad 2 = 2y \quad \Rightarrow \quad y = 1$

$$I = \int_0^2 \frac{x^2}{\sqrt{2-x}}\ dx = \int_0^1 \frac{4y^2}{\sqrt{2-2y}}\ 2dy$$

$$= 4\sqrt{2} \int_0^1 y^2\ (1-y)^{-\frac{1}{2}}\ dy$$

$$= 4\sqrt{2}\ \beta\left(3, \frac{1}{3}\right) = 4\sqrt{2}\ \frac{\Gamma(3)\ \Gamma(1/2)}{\Gamma(7/2)}$$

$$= 4\sqrt{2}\ \frac{2!\ \Gamma(1/2)}{(5/2)\ (3/2)\ (1/2)\ \Gamma(1/2)} = \frac{64\sqrt{2}}{15}$$

CamScanner

EXAMPLE 3▶

Evaluate $\int_0^a y^4 \sqrt{a^2 - y^2}\, dy$

SOLUTION

Let $y^2 = a^2 x \quad \Rightarrow \quad y = a\sqrt{x}$

$$dy = a\frac{1}{2}x^{-\frac{1}{2}}\, dx$$

$$I = \int_0^a y^4 \sqrt{a^2 - y^2}\, dy = \frac{a}{2}\int_0^1 (a^2 x)^2 \sqrt{a^2 - a^2 x}\; x^{-1/2}\, dx$$

$$= \frac{a^6}{2}\int_0^1 x^{3/2}(1-x)^{1/2}\, dx$$

$$= \frac{a^6}{2}\beta\left(\frac{5}{2}, \frac{3}{2}\right) = \frac{a^6}{2}\frac{\Gamma(5/2)\,\Gamma(3/2)}{\Gamma(4)}$$

$$= \frac{a^6}{2}\frac{[(3/2)(1/2)\Gamma(1/2)][(1/2)\Gamma(1/2)]}{3!} = \frac{a^6\,\pi}{16}$$

EXAMPLE 4▶

Evaluate $\int_0^{\pi/2} \sin^4\theta \cos^5\theta\, d\theta$

SOLUTION

$$2m-1=4\ ,\ 2n-1=5 \quad \Rightarrow m=\frac{5}{2}\ ,\ n=3$$

$$I = \frac{1}{2}\beta\left(\frac{5}{2}, 3\right) = \frac{\Gamma(5/2)\,\Gamma(3)}{2\Gamma(11/2)}$$

CamScanner

$$= \frac{\Gamma(5/2)\,(2)\,(1)}{(2)(9/2)(7/2)(5/2)\,\Gamma(5/2)} = \frac{8}{315}$$

EXAMPLE 5►

Evaluate $\int_0^{\frac{\pi}{2}} \sin^6\theta \; d\theta$

SOLUTION

$$2m-1=6 \;,\; 2n-1=0 \quad \Rightarrow m=\frac{7}{2} \quad , \quad n=\frac{1}{2}$$

$$I = \int_0^{\frac{\pi}{2}} \sin^6\theta \; d\theta = \beta\left(\frac{7}{2}, \frac{1}{2}\right) = \frac{\Gamma\left(\frac{7}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma(4)}$$

$$= \frac{(5/2)(3/2)(1/2)\,\pi}{(2)(3)(2)(1)} = \frac{5\pi}{32}$$

EXAMPLE 6►

Evaluate $\int_0^{\pi} \cos^4\theta \; d\theta$

SOLUTION

$$I = \int_0^{\pi} \cos^4\theta \; d\theta = 2\int_0^{\frac{\pi}{2}} \cos^4\theta \; d\theta$$

$$2m-1=0 \;,\; 2n-1=4 \;\Rightarrow\; m=\frac{1}{2} \;,\; n=\frac{5}{2}$$

CamScanner

$$I=\frac{2}{2}\beta\left(\frac{1}{2},\frac{5}{2}\right)=\frac{2}{2}\frac{\Gamma(1/2)\,(3/2)\,(1/2)\,\Gamma(1/2)}{\Gamma(3)}=\frac{3}{8}\pi$$

EXAMPLE 7►

Prove that $\int_0^{\pi/2}\sqrt{\tan t}\;dt=\frac{\pi}{\sqrt{2}}$.

SOLUTION:

$$I=\int_0^{\pi/2}\sqrt{\tan t}\;dt=\int_0^{\pi/2}\sin^{\frac{1}{2}}t\cos^{-\frac{1}{2}}t\;dt$$

$$2m-1=\frac{1}{2}\,,\;2n-1=-\frac{1}{2}\;\Rightarrow m=\frac{3}{4}\,,n=\frac{1}{4}$$

$$I=\frac{1}{2}\beta\left(\frac{3}{4},\frac{1}{4}\right)=\frac{1}{2}\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma(1)}=\frac{1}{2}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)$$

Using eq. (4)

$$I=\frac{1}{2}\frac{\pi}{\sin\frac{\pi}{4}}=\frac{\pi}{\sqrt{2}}$$

EXAMPLE 8►

Prove that $\int_0^1 x\left(1-x^3\right)^{\frac{1}{3}}dx=\frac{2\sqrt{3}\,\pi}{27}$.

SOLUTION

Let $x^3=y\;\Rightarrow\;x=y^{\frac{1}{3}}\;\Rightarrow\;dx=\frac{1}{3}y^{-\frac{2}{3}}dy$

CamScanner